

Random triangles in planar regions containing a fixed point^{*}

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Abstract

In this article we provide several exact formulae to calculate the probability that a random triangle chosen within a planar region contains a given fixed point O in the interior of that region or on its boundary. These formulae are in terms of only one integration over the boundary of the region of an appropriate function with respect to a distribution which depends of the point O . One of our formulae is a generalization of the formula in Proposition II ([3], page 226). We use these formulae to show that this probability is always less than or equal to $1/4$ and this maximum is attained if and only if the region is symmetric with respect to O . We recover the known probability in the case of an equilateral triangle and its center of mass: $\frac{2}{27} + 20\frac{\ln 2}{81}$ ([3], [8] and [9]). We compute this probability in the case of a regular polygon and its center of mass for the point O . Some upper bounds are provided. Other families of regions are studied. In the case of perhaps the simplest applications, the family of Limaçons $r = a + \cos t$, $a > 1$, and O the origin of the polar coordinates, the probability is $\frac{1}{4} - \frac{12a^2(4a^2+1)}{(2a^2+1)^3\pi^2}$.

1 Introduction

Geometric probabilities are a rich source of beautiful examples in analysis when calculations happen to add up in a neat way (see [2], [4], [6], and [7]). This study is also such an instance. In what follows we are concerned with the question: *What is the probability that choosing three points at random A , B and C inside of a region \mathbf{R} (uniform distribution with respect*

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If A is an arbitrary point in \mathbf{R} , we let $s = s(A)$ its projection from O on the boundary (Figure 1 (b)). A natural question here is: *what is the distribution of the points $s = s(A)$ on the boundary?*

We will let this distribution be f , i.e., the function $f : \partial\mathbf{R} \rightarrow [0, 1]$ satisfies:

$$\int_{\widehat{st}} f(x) dx = \frac{1}{2} \int_{\theta_s}^{\theta_t} r(\theta)^2 d\theta, \quad \theta_s \leq \theta_t, \theta_t - \theta_s \leq 2\pi, \quad (1)$$

for every two points on $s = (r(\theta_s), \theta_s)$ and $t = (r(\theta_t), \theta_t)$ (in polar) on $\partial\mathbf{R}$, with the understanding that the arc \widehat{st} is the oriented arc along the boundary from s to t and the orientation is always counterclockwise. By our assumption $\int_{\partial\mathbf{R}} f(x) dx = 1$. We will see that our method

works basically for any parametrization of the boundary $s \rightarrow \gamma(s)$ and a distribution g such that for which $g(s)|\gamma'(s)|ds = \frac{1}{2}r(\theta_\gamma(s))^2 d\theta_\gamma(s)$.

For a point \mathbf{t} on the boundary let us denote by $F(\mathbf{t})$ its projection through O on the boundary. Clearly $F(F(t)) = t$ for every $t \in \partial\mathbf{R}$, so F is an involution on $\partial\mathbf{R}$. Suppose we fix a point \mathbf{u} on the boundary and introduce two new functions:

$$G(t) = \int_{\widehat{ut}} f(x) dx \quad \text{and} \quad H(t) = G(F(t)) - G(t), \quad t \in \widehat{uv}$$

where $v = F(u)$. The geometric interpretation of $H(t)$ is clearly the area of part of the region \mathbf{R} which is to the right of the line $tF(t)$ (looking from t to $F(t)$, see the green shaded area in Figure 1 (b)).

The following formulae are of primary importance in this work.

THEOREM 1.1. (i) *The probability that choosing three points at random A , B and C inside of a region \mathbf{R} of area 1 (with uniform distribution with respect to the area), the triangle $\triangle ABC$ contains the fixed interior point O of the region \mathbf{R} is given by*

$$\mathcal{P} = -G(\mathbf{v})^2[3 - 2G(\mathbf{v})] + 6 \int_{\widehat{uv}} H(t) [1 - H(t)] f(t) dt, \quad (2)$$

where $\mathbf{v} = F(\mathbf{u})$.

(ii) *The point $\mathbf{u} \in \partial\mathbf{R}$ can be chosen in such a way the formula (2) reduces to*

$$\mathcal{P} = \frac{1}{4} - 6 \int_{\widehat{uv}} \left[\frac{1}{2} - H(t) \right]^2 f(t) dt. \quad (3)$$

(iii) *Independent of the point u , one can also write*

$$\mathcal{P} = -\frac{1}{2} + 3 \int_{\partial\mathbf{R}} H(t) [1 - H(t)] f(t) dt = \frac{1}{4} - 3 \int_{\partial\mathbf{R}} \left[\frac{1}{2} - H(t) \right]^2 f(t) dt \quad (4)$$

For polygonal regions the two functions H and f are not difficult to compute. For other types of regions one may want to use polar coordinates for the computations of these functions or intrinsic parameterizations. For instance, if the curve is the limaçon given by the polar equation $r = \frac{1}{3}\sqrt{\frac{2}{\pi}}(2 + \cos\theta)$, $\theta \in [0, 2\pi]$, then the integral in (3) reduces to a pretty straightforward calculation $\mathcal{P} = \frac{1}{4} - \frac{272}{243\pi^2}$. Let us point out that one can arrive at the first formula in (4) by a simple probabilistic argument as described in ([3]).

2 Proof of Theorem 1.1

For two points on the boundary s and t , we denote by \mathbf{R}_{st} the region determined by intersection of \mathbf{R} with the interior of the angle $\angle sOt$. This region \mathbf{R}_{st} , is always a convex region. If we consider two arbitrary points on the boundary (Figure 1 (b)), say s and t , then it is clear that the third point C must be in the region \mathcal{R}_{st} determined by intersection of the square with the angle $\angle F(s)OF(t)$ (colored blue in Figure 2), in order for the point O to be inside of $\triangle ABC$, for every points A and B on the segments \overline{Os} and \overline{Ot} , respectively. Hence, the probability we are looking for is equal to

$$\mathcal{P}(O \in \triangle ABC) = \int_{\mathbf{R}} \int_{\mathbf{R}} \int_{\mathbf{R}} \chi_{O \in \triangle ABC}(z_A, z_B, z_C) dz_A dz_B dz_C.$$

where dz_A , dz_B and dz_C are the elements of area associated to the complex variable for each of the points A , B and C in \mathbf{R} and $\chi_E(\cdot)$ is the characteristic function of the set E , defined as usual by

$$\chi_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{if } \omega \notin E. \end{cases}$$

As we observed earlier, $\chi_{O \in \triangle ABC}(z_A, z_B, z_C) = \chi_{O \in \triangle s(A)BC}(z_A, z_B, z_C)$, which means that we can change to polar coordinates in the integration over \mathbf{R} with respect to the variable z_A :

$$\int_{\mathbf{R}^3} \chi_{O \in \triangle ABC}(z_A, z_B, z_C) dz_A dz_B dz_C = \int_{\mathbf{R}^2} \left[\int_0^{2\pi} \int_0^{r(\theta)} \chi_{O \in \triangle s(A)BC}(re^{i\theta}, z_B, z_C) r dr d\theta \right] dz_B dz_C.$$

In the integrant above $\chi_{O \in \triangle s(A)BC}(re^{i\theta}, z_B, z_C) = \chi_{O \in \triangle s(A)BC}(s(A), z_B, z_C)$ which means this is a constant function along the segment $Os(A)$. This results in the integration with respect to r to be $\frac{r(\theta)^2}{2}$. Using the definition of the distribution f along the boundary, we can change the variable from θ to s , $\frac{r(\theta)^2}{2} d\theta = f(s) ds$, and obtain that

$$\mathcal{P} = \int_{\mathbf{R}^2} \left[\int_{\partial \mathbf{R}} \chi_{O \in \triangle sBC}(s, z_B, z_C) f(s) ds \right] dz_B dz_C.$$

Doing the same thing for the variable B , we end up with

$$\mathcal{P} = \int_{\mathbf{R}} \left[\int_{\partial \mathbf{R}} \int_{\partial \mathbf{R}} \chi_{O \in \Delta_{st}C}(s, t, z_C) f(s) f(t) dt ds \right] dz_C.$$

For s and t fixed on the boundary,

$$\int_{\mathbf{R}} \chi_{O \in \Delta_{st}C}(s, t, z_C) dz_C = \text{Area}(\mathcal{R}_{st})$$

so, interchanging the integrals and integrating with respect to z_C first, we obtain

$$\mathcal{P} = \int_{\partial \mathbf{R}} \int_{\partial \mathbf{R}} \text{Area}(\mathcal{R}_{st}) f(s) f(t) ds dt. \quad (5)$$

From now on, we are going to manipulate this to bring it in the form of (2). We could have started with this representation, which is basically the first intuitive reduction and the purpose of introducing the distribution f on the boundary. For a $t \in \partial \mathbf{R}$ fixed, and $s \in tF(t)$, the area of \mathcal{R}_{st} is given by $\int_{F(t)F(s)} f(x) dx$. If $s \in F(t)t$, the area of \mathcal{R}_{st} is given by

$\int_{tF(s)} f(x) dx$ (we will skip the arc notation for simplicity).

In (5), we split the boundary into two parts $\partial \mathbf{R} = \widehat{\mathbf{uv}} \cup \widehat{\mathbf{vu}}$: and the corresponding integrals $\mathcal{P} = I_1 + I_2$.

Then, the inner integration we split into four parts (Figure 2),

$$\begin{aligned} I_1 &= \int_{\widehat{\mathbf{uv}}} \left[\int_{\widehat{\mathbf{ut}}} (G(F(t)) - G(F(s))) f(s) ds + \int_{\widehat{\mathbf{tv}}} (G(F(s)) - G(F(t))) f(s) ds + \right. \\ &\quad \left. + \int_{\widehat{\mathbf{vF(t)}}} (1 + G(F(s)) - G(F(t))) f(s) ds + \int_{\widehat{\mathbf{F(t)F(v)}}} (G(F(t)) - G(F(s))) f(s) ds \right] f(t) dt = J_1 + J_2. \end{aligned}$$

The last integration was split into two integrals as follows

$$\begin{aligned} J_1 &= \int_{\widehat{\mathbf{uv}}} \left[- \int_{\widehat{\mathbf{ut}}} G(F(s)) f(s) ds + \int_{\widehat{\mathbf{tv}}} G(F(s)) f(s) ds + \right. \\ &\quad \left. + \int_{\widehat{\mathbf{vF(t)}}} G(F(s)) f(s) ds - \int_{\widehat{\mathbf{F(t)F(v)}}} G(F(s)) f(s) ds \right] f(t) dt, \text{ and} \\ J_2 &= \int_{\widehat{\mathbf{uv}}} \left[\int_{\widehat{\mathbf{ut}}} G(F(t)) f(s) ds - \int_{\widehat{\mathbf{tv}}} G(F(t)) f(s) ds + \right. \\ &\quad \left. + \int_{\widehat{\mathbf{vF(t)}}} (1 - G(F(t))) f(s) ds + \int_{\widehat{\mathbf{F(t)F(v)}}} G(F(t)) f(s) ds \right] f(t) dt. \end{aligned}$$

The second integral J_2 can be simplified to

$$\begin{aligned}
J_2 &= \int_{\mathbf{uv}} [G(F(t))G(t)ds - G(F(t))(G(\mathbf{v}) - G(t)) + \\
&+ (1 - G(F(t)))(G(F(t)) - G(\mathbf{v})) + G(F(t))(1 - G(F(t)))] f(t)dt = \\
&= \int_{\mathbf{uv}} [2G(F(t))(1 - G(F(t)) + G(t)) - G(\mathbf{v})] f(t)dt = \\
&= \int_{\mathbf{uv}} 2G(F(t)) [1 - G(F(t)) + G(t)] f(t)dt - G(\mathbf{v})^2.
\end{aligned}$$

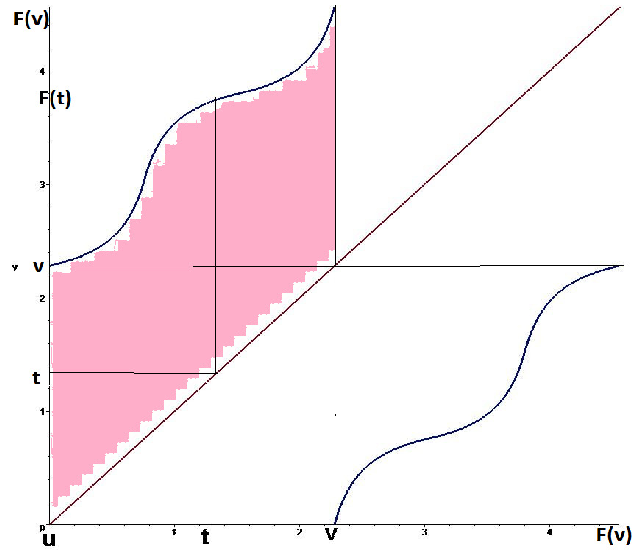


Figure 2, Graph of F from Section 5

For J_1 we will interchange integration order of integration in each one of those four integrals (Figure 2), for instance,

$$\begin{aligned}
\int_{\mathbf{uvut}} G(F(s))f(s)f(t)dsdt &= \int_{\mathbf{uvsv}} G(F(s))f(t)f(s)dtds = \\
&= \int_{\mathbf{uv}} G(F(s))[G(\mathbf{v}) - G(s)]f(s)ds.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\int_{\mathbf{uvtv}} G(F(s))f(s)f(t)dsdt &= \int_{\mathbf{uvus}} G(F(s))f(t)f(s)dtds = \\
&= \int_{\mathbf{uv}} G(F(s))G(s)f(s)ds, \\
\int_{\mathbf{uvvF(t)}} G(F(s))f(s)f(t)dsdt &= \int_{\mathbf{vuF(s)v}} G(F(s))f(t)f(s)dtds = \\
&= \int_{\mathbf{vu}} G(F(s))[G(\mathbf{v}) - G(F(s))]f(s)ds,
\end{aligned}$$

and finally

$$\begin{aligned} \int_{\mathbf{u}\mathbf{F}(\mathbf{t})\mathbf{F}(\mathbf{v})} \int G(F(s))f(s)f(t)dsdt &= \int_{\mathbf{v}\mathbf{u}\mathbf{F}(\mathbf{s})} \int G(F(s))f(t)f(s)dtds = \\ &= \int_{\mathbf{v}\mathbf{u}} G(F(s))G(F(s))f(s)ds. \end{aligned}$$

Putting all these last four integrals together, gives

$$J_1 = \int_{\mathbf{u}\mathbf{v}} G(F(s))[2G(s) - G(\mathbf{v})]f(s)ds + \int_{\mathbf{v}\mathbf{u}} G(F(s))[G(\mathbf{v}) - 2G(F(s))]f(s)ds.$$

Hence, adding J_1 and J_2 one obtains

$$\begin{aligned} I_1 &= \int_{\mathbf{u}\mathbf{v}} G(F(t)) [2 - 2G(F(t)) + 4G(t) - G(\mathbf{v})] f(t)dt + \\ &+ \int_{\mathbf{v}\mathbf{u}} G(F(t))[G(\mathbf{v}) - 2G(F(t))]f(t)dt - G(\mathbf{v})^2. \end{aligned} \tag{6}$$

Now, we handle I_2 the same way

$$\begin{aligned} I_2 &= \int_{\mathbf{v}\mathbf{u}} \left[\int_{\mathbf{u}\mathbf{F}(\mathbf{t})} (G(F(s)) - G(F(t)))f(s)ds + \int_{F(t)\mathbf{v}} (1 + G(F(t)) - G(F(s)))f(s)ds + \right. \\ &+ \left. \int_{\mathbf{v}\mathbf{t}} (G(F(t)) - G(F(s)))f(s)ds + \int_{\mathbf{t}\mathbf{F}(\mathbf{v})} (G(F(s)) - G(F(t)))f(s)ds \right] f(t)dt = K_1 + K_2. \end{aligned}$$

As before, we define the terms K_1 and K_2 in the following way:

$$\begin{aligned} K_1 &= \int_{\mathbf{v}\mathbf{u}} \left[\int_{\mathbf{u}\mathbf{F}(\mathbf{t})} G(F(s))f(s)ds - \int_{\mathbf{F}(\mathbf{t})\mathbf{v}} G(F(s))f(s)ds + \right. \\ &- \left. \int_{\mathbf{v}\mathbf{t}} G(F(s))f(s)ds + \int_{\mathbf{t}\mathbf{F}(\mathbf{v})} G(F(s))f(s)ds \right] f(t)dt, \\ K_2 &= \int_{\mathbf{v}\mathbf{u}} \left[\int_{U\mathbf{F}(t)} -G(F(t))f(s)ds + \int_{\mathbf{F}(\mathbf{t})\mathbf{v}} (1 + G(F(t)))f(s)ds + \right. \\ &+ \left. \int_{\mathbf{v}\mathbf{t}} G(F(t))f(s)ds - \int_{\mathbf{t}\mathbf{F}(\mathbf{v})} G(F(t))f(s)ds \right] f(t)dt. \end{aligned}$$

Let's start with the easy part K_2 , and observe that

$$\begin{aligned} K_2 &= \int_{\mathbf{v}\mathbf{u}} [-G(F(t))G(F(t)) + (1 + G(F(t)))(G(\mathbf{v}) - G(F(t))) + \\ &+ G(F(t))(G(t) - G(\mathbf{v})) - G(F(t))(1 - G(t))]f(t)dt \Rightarrow \end{aligned}$$

$$K_2 = \int_{\mathbf{v}\mathbf{u}} [-2G(F(t))[1 - G(t) + G(F(t))]f(t)dt + G(\mathbf{v})(1 - G(\mathbf{v})).$$

For the first term in K_1 we have

$$\begin{aligned} \int \int_{\mathbf{v}\mathbf{u}\mathbf{F}(\mathbf{t})} G(F(s))f(s)f(t)dsdt &= \int \int_{\mathbf{u}\mathbf{v}\mathbf{F}(\mathbf{s})} G(F(s))f(t)f(s)dtds = \\ &= \int_{\mathbf{u}\mathbf{v}} G(F(s))(1 - G(F(s)))f(s)ds. \end{aligned}$$

For the second term, we get

$$\begin{aligned} \int \int_{\mathbf{v}\mathbf{u}\mathbf{F}(\mathbf{t})\mathbf{v}} G(F(s))f(s)f(t)dsdt &= \int \int_{\mathbf{u}\mathbf{v}\mathbf{F}(\mathbf{s})} G(F(s))f(t)f(s)dtds = \\ &= \int_{\mathbf{u}\mathbf{v}} G(F(s))(G(F(s)) - G(\mathbf{v}))f(s)ds. \end{aligned}$$

Then, the third term

$$\begin{aligned} \int \int_{\mathbf{v}\mathbf{u}\mathbf{t}} G(F(s))f(s)f(t)dsdt &= \int \int_{\mathbf{v}\mathbf{u}\mathbf{t}} G(F(s))f(t)f(s)dtds = \\ &= \int_{\mathbf{v}\mathbf{u}} G(F(s))(1 - G(t))f(s)ds, \end{aligned}$$

and finally

$$\begin{aligned} \int \int_{\mathbf{v}\mathbf{u}\mathbf{t}} G(F(s))f(s)f(t)dsdt &= \int \int_{\mathbf{v}\mathbf{u}\mathbf{t}} G(F(s))f(t)f(s)dtds = \\ &= \int_{\mathbf{v}\mathbf{u}} G(F(s))(G(t) - G(\mathbf{v}))f(s)ds. \end{aligned}$$

Hence, all these together gives

$$K_1 = \int_{\mathbf{u}\mathbf{v}} G(F(s))(1 - 2G(F(s)) + G(\mathbf{v}))f(s)ds + \int_{\mathbf{v}\mathbf{u}} G(F(s))(2G(t) - 1 - G(\mathbf{v}))f(s)ds.$$

This gives the second integral I_2 :

$$\begin{aligned} I_2 &= \int_{\mathbf{u}\mathbf{v}} G(F(s))(1 - 2G(F(s)) + G(\mathbf{v}))f(s)ds + \\ &\int_{\mathbf{v}\mathbf{u}} G(F(s))(4G(t) - 3 - 2G(F(t)) - G(\mathbf{v}))f(s)ds + G(\mathbf{v})(1 - G(\mathbf{v})), \end{aligned}$$

and so

$$\boxed{
\begin{aligned}
\mathcal{P} = & \int_{\mathbf{uv}} G(F(t)) [3 - 4G(F(t)) + 4G(t)] f(t) dt - \\
& - \int_{\mathbf{vu}} G(F(t)) [3 + 4G(F(t)) - 4G(t)] f(t) dt + G(\mathbf{v})(1 - 2G(\mathbf{v})).
\end{aligned}
} \tag{7}$$

Next, we observe that $H'(t) = f(F(t))F'(t) - f(t)$, and so the formula (7) can be written as, after a change of variables in the second integral $t = F(t')$,

$$\begin{aligned}
\mathcal{P} = & \int_{\mathbf{uv}} G(F(t)) [3 - 4H(t)] f(t) dt - \\
& - \int_{\mathbf{uv}} G(t) [3 - 4H(t)] f(F(t))F'(t) dt + G(\mathbf{v})(1 - 2G(\mathbf{v})) \Leftrightarrow \\
\mathcal{P} = & \int_{\mathbf{uv}} H(t) [3 - 4H(t)] f(t) dt - \\
& - \int_{\mathbf{uv}} G(t) [3 - 4H(t)] H'(t) dt + G(\mathbf{v})(1 - 2G(\mathbf{v})).
\end{aligned}$$

An integration by parts in the last integral gives

$$\int_{\mathbf{uv}} G(t) [3 - 4H(t)] H'(t) dt = (3H(t) - 2H(t)^2)G(t)|_{\mathbf{u}}^{\mathbf{v}} - \int_{\mathbf{uv}} H(t) [3 - 2H(t)] f(t) dt$$

Since $H(\mathbf{u}) = G(\mathbf{v})$ and $H(\mathbf{v}) = 1 - G(\mathbf{v})$, we obtain (2). The function $H(t)$ (which under our assumptions must be continuous) we see that this function must take the value of $1/2$ by the intermediate value theorem. Indeed, $H(t) + H(F(t)) = 1$ and then if for some point $H(t_0) < 1/2$ then $H(F(t_0)) > 1/2$ which means somewhere, H it must take the value $1/2$. Then we take that point to be \mathbf{u} . So the first term in formula (2) is equal to $-1/2$, which shows (3) is true.

To obtain (4), we apply (2) for u , then for $v = F(u)$ instead of u , and add the two identities up:

$$2\mathcal{P} = -(Q(G(v)) + Q(1 - G(v))) + 6 \int_{\partial \mathbf{R}} H(t) [1 - H(t)] f(t) dt,$$

where $Q(x) = x^2(3 - 2x)$. Since $Q(x) + Q(1 - x) = 1$, we get (4). ■

3 A few consequences

Let us start with the second important result of this paper. We will say that a region \mathbf{R} is symmetric with respect to O if $r(\theta) = r(\theta + \pi)$ for all $\theta \in [0, 2\pi]$.

THEOREM 3.1. *The probability discussed in Theorem 1.1 satisfies $\mathcal{P} \leq 1/4$. Moreover, $\mathcal{P} = 1/4$ if and only if \mathbf{R} is symmetric.*

Proof. The first claim follows from (3). If the region is symmetric with respect to O then $H(t) = G(F(t)) - G(t) = 1/2$ for all $t \in \mathbf{uv}$. By (3), we have $\mathcal{P} = \frac{1}{4}$. Conversely, if $\mathcal{P} = \frac{1}{4}$ then $\int_{\mathbf{uv}} \left[\frac{1}{2} - H(t)\right]^2 f(t) dt = 0$ which attracts $H(t) = 1/2$ almost everywhere, but since we are dealing with at least continuous functions, we must have $H(t) = 1/2$ for all t on the boundary. Hence $r(\theta + \pi)^2 = r(\theta)^2$ which implies that \mathbf{R} is symmetric with respect to O . \square

COROLLARY 3.2. *Suppose the point O approaches a point \mathbf{u} on the boundary of \mathbf{R} such that the region \mathbf{R} is on one side of a line through \mathbf{u} . Then*

$$\lim_{O \rightarrow \partial \mathbf{R}} \mathcal{P}_O = 0.$$

Proof. If O gets close to \mathbf{u} on the boundary, there exists a line containing O such that it cuts the region \mathbf{R} into two regions, one of which has an area that goes to 0 as O gets closer to u . We can accomplish this by taking a parallel through O to the line in the hypothesis of the corollary since the region \mathbf{R} is bounded. We take that line as \mathbf{uv} . Then by (2), we have $\mathcal{P}_O \leq \frac{3}{2}G(\mathbf{v}) \rightarrow 0$. \square

We observe that this limiting behavior is not satisfied for every point on the boundary \mathbf{R} (see Section 4).

In [3], Halász and Kleitman studied the maximum of \mathcal{P} as a function of O for a triangle. They show that this maximum is attained for the center of mass of the triangle. So, it is interesting to ask the similar problem:

Where is position of O to have a maximum of \mathcal{P}_O for a region fixed region \mathbf{R} ?

Theorem 3.1 provides an answer to this question if \mathbf{R} is symmetric but what happens in general? The formula (3) suggests that \mathcal{P} is maximum where the function $t \rightarrow (1/2 - H(t))$ has the smallest L^2 norm with respect to this measure $f(x)dx$ on the boundary. On the other hand this measure depends on the point O too, and as O moves toward a point at which $H(t)$ is closer to $1/2$ the density f on \mathbf{uv} gets bigger. So, the trade off is not clear and the problem can be decided only if a more precise expression for \mathcal{P} is found. However, the formula (2) provides a better upper bound in the case of non-symmetric domains.

THEOREM 3.3. *The probability discussed in Theorem 1.1 satisfies*

$$\mathcal{P} \leq \frac{1}{4} - 2 \left(\frac{1}{2} - h \right)^3 \quad (8)$$

where $h = \min\{H(t) | t \in \partial \mathbf{R}\}$.

Proof. We observe that $H(t)(1 - H(t)) \leq \frac{1}{4}$ for every t . Hence, using (2), we get

$$\mathcal{P} \leq \frac{6}{4}G(\mathbf{v}) - G(\mathbf{v})^2[3 - 2G(\mathbf{v})] = \frac{x(3 - 6x + 4x^2)}{2},$$

where $x = G(\mathbf{v})$. Since \mathbf{u} is arbitrary, this inequality is true for the smallest value of the function

$$W(x) = \frac{x(3 - 6x + 4x^2)}{2} = \frac{1}{4} - \frac{(1 - 2x)^3}{4} = \frac{1}{4} - 2\left(\frac{1}{2} - x\right)^3$$

over the interval $[h, 1/2]$ (we know that x can be $1/2$). The derivative of this function is $W'(x) = \frac{3(1-2x)^2}{2} \geq 0$ and so W is strictly increasing. This gives a minimum of W at h . Therefore, (8) follows from these simple considerations. \blacksquare

Another observation here is that H has a minimum at a point t_0 where $H'(t_0) = 0$. If we use polar representation, this means $r(t_0 + \pi) = r(t_0)$. In other words, O divides the segment $t_0 F(t_0)$ into equal parts.

For \mathbf{R} a triangle and if O is the center of mass, Theorem 3.3 shows that $\mathcal{P} \leq \frac{182}{729}$.

4 A family of Limaçons between a circle and a Cardioid

Perhaps the simplest curve that fits as the boundary of a region in our framework and for which all the involved calculations are quite elementary is the case of a Limaçon. We will start with the equations in polar coordinates $r = \frac{\sqrt{2}}{\sqrt{(2a^2+1)\pi}}(a + \cos t)$, $t \in [0, 2\pi]$. This is, in fact, a family of Limaçons over the parameter $a \in (1, \infty)$. In Figure 3 we have included these curves for $a \in \{10, 3, 2, 1.5, 1\}$. Let us remark that $a = 1$ is in fact a Cardioid and the point O is on the boundary in this case. At the other extreme, if $a \rightarrow \infty$ we get a circle

centered at the origin of radius $\frac{1}{\sqrt{\pi}}$ (area 1).

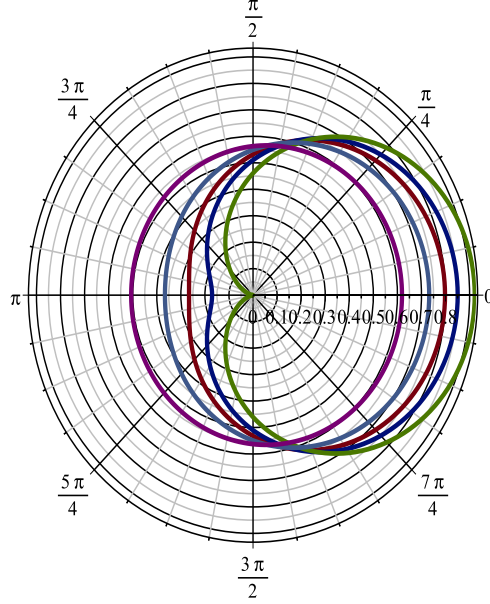


Figure 3

We observe first that the area of each such curve is 1:

$$Area = \frac{1}{2} \int_0^{2\pi} r(t)^2 dt = \frac{1}{(2a^2 + 1)\pi} \int_0^{2\pi} (a^2 + 2a \cos t + \frac{1 + \cos 2t}{2}) dt = 1.$$

So, the distribution on the boundary can be taken to be $\frac{1}{2}r(t)^2 dt$. Then the function $H(t) = \int_t^{t+\pi} \frac{1}{2}r(t)^2 dt$ or

$$H(t) = \frac{1}{(2a^2 + 1)\pi} \int_t^{t+\pi} (a^2 + 2a \cos t + \frac{1 + \cos 2t}{2}) dt = \frac{a^2\pi - 4a \sin t + \pi/2}{(2a^2 + 1)\pi} \Rightarrow$$

$$\frac{1}{2} - H(t) = \frac{4a \sin t}{(2a^2 + 1)\pi}, t \in [0, \pi].$$

Then, since $H(0) = 1/2$, formula (3) gives

$$\mathcal{P} = \frac{1}{4} - 6 \frac{16a^2}{(2a^2 + 1)^3 \pi^3} \int_0^\pi (a^2 + 2a \cos t + \frac{1 + \cos 2t}{2}) \sin^2 t dt.$$

Since $(a^2 + 2a \cos t + \frac{1 + \cos 2t}{2}) \sin^2 t = a^2 \frac{1 - \cos 2t}{2} + \frac{2a}{3} \frac{d}{dt}(\sin^3 t) + \frac{1 - \cos^2 2t}{4}$ the above integration leads to

$$\mathcal{P}_a = \frac{1}{4} - 6 \frac{16a^2}{(2a^2 + 1)^3 \pi^3} \left(\frac{a^2\pi}{2} + \frac{\pi}{8} \right) = \frac{1}{4} - \frac{12a^2(4a^2 + 1)}{(2a^2 + 1)^3 \pi^2}.$$

Interestingly enough as $a \rightarrow 1$ we get $\mathcal{P}_1 = \frac{1}{4} - \frac{20}{9\pi^2} \approx 0.0248418142$. So, it is possible to have the probability \mathcal{P}_O strictly positive for a point O on the boundary of \mathbf{R} . We observe that

$$h = \min H(t) = \frac{a^2\pi - 4a + \pi/2}{(2a^2 + 1)\pi} = \frac{1}{2} - \frac{4a}{(2a^2 + 1)\pi}.$$

This checks out the inequality (8) which reduces in this case to the obvious inequality $\pi \geq \frac{32}{3(4a^2+1)}$ for $a > 1$.

5 The case when \mathbf{R} is an equilateral triangle

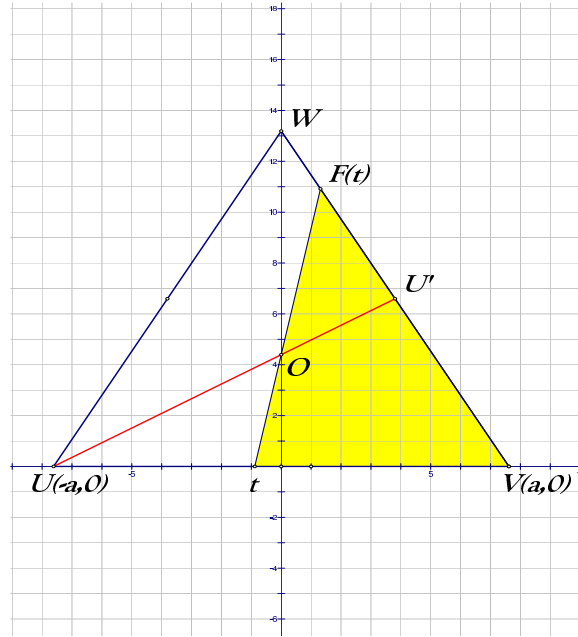


Figure 4

Let us use formula (3) to compute this probability for an equilateral triangle. We will do the calculations for the triangle positioned as in Figure 4, with vertices $U(-a, 0)$, $V(a, 0)$ and $W(0, a\sqrt{3})$. The area of this triangle is $a^2\sqrt{3}$ so we will take $a = 3^{-1/4}$ in order for the area to be equal to 1. Since UU' is a line of symmetry, we can use formula (3). The distributions on all sides are all equal: $f \equiv \frac{a\sqrt{3}}{6} = 3^{-3/4}/2$. To compute F let us consider $t \in [-a, 0]$ and observe that the line tO has equation $X(\frac{a\sqrt{3}}{3} - 0) - Y(0 - t) - \frac{at\sqrt{3}}{3} = 0$ and the line VW has equation $X/a + Y/(a\sqrt{3}) = 1$. Solving this system gives us $F(t) = (\frac{-2at}{a-3t}, \frac{a(a-t)\sqrt{3}}{a-3t})$. This implies that $F(t)V = \frac{2a(a-t)}{a-3t}$. Hence,

$$H(t) = \frac{a(a-t)^2}{a-3t} \sin 60^\circ = \frac{a(a-t)^2\sqrt{3}}{2(a-3t)}$$

for $t \in [-a, 0]$. Clearly, for $t \in [0, a]$ we have $H(t) = 1 - \frac{a(a+t)^2\sqrt{3}}{2(a+3t)}$ and so the function $H(t)(1 - H(t))$ is symmetric with respect to each vertex and midpoint on the boundary ∂R . As a result the calculation in (3) reduces to

$$\mathcal{P} = \frac{1}{4} - 6 \int_U^{U'} \left(\frac{1}{2} - H(t) \right)^2 f(t) dt = \frac{1}{4} - 6 \cdot 3 \frac{a\sqrt{3}}{6} \int_{-a}^0 \left(\frac{1}{2} - H(t) \right)^2 dt,$$

Changing the variable in the last integration $s = (a - t)/a$, $ds = -dt/a$, $s \in [1, 2]$, we obtain

$$\mathcal{P} = \frac{1}{4} - 3a^2\sqrt{3} \int_1^2 \left[\frac{1}{2} - \frac{s^2}{2(3s-2)} \right]^2 ds = \frac{1}{4} - \frac{3}{4} \int_1^2 \frac{(2-s)^2(s-1)^2}{(3s-2)^2} ds.$$

Changing the variable again, $3s - 2 = x$, the partial fraction decomposition comes easily:

$$\begin{aligned} \mathcal{P} &= \frac{1}{4} - \frac{1}{324} \int_1^4 \frac{(4-x)^2(x-1)^2}{x^2} dx = \frac{1}{4} - \frac{1}{324} \int_1^4 (x^2 - 10x + 33 - \frac{40}{x} + \frac{16}{x^2}) dx = \\ &= \frac{1}{4} - \frac{1}{324} \left(\frac{x^3}{3} - 5x^2 - \frac{16}{x} + 40 \ln x \right) \Big|_1^4 = \frac{2(3+10 \ln 2)}{81} \approx 0.2452215261. \end{aligned}$$

This is the same answer obtained in [8].

6 Regular Polygon with $2m + 1$ vertices

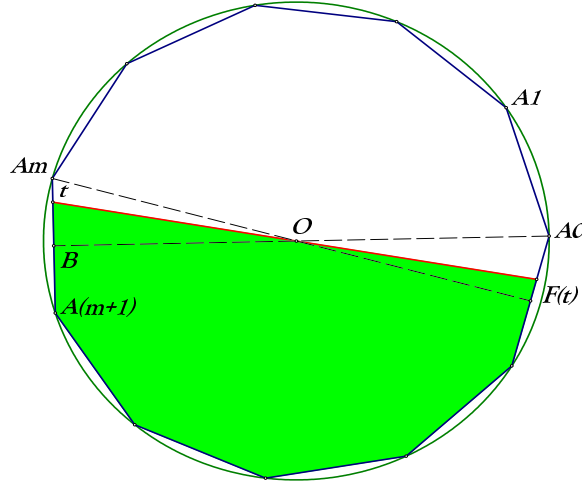


Figure 5

Let us generalize the previous situation to a regular polygon with $2m + 1$ vertices ($m \geq 1$). We will refer to Figure 5, where the regular polygon is centered at the origin and it has vertices $A_k = Re^{i\theta}$ where $\theta_k = \frac{2k\pi}{2m+1}$, $k = 0, 1, \dots, 2m$. Here R is chosen so that the area of the polygon is equal to 1, i.e., $R = \sqrt{\frac{2}{(2m+1) \sin \frac{2\pi}{2m+1}}}$. The side lengths are $\ell = 2R \sin(\theta_1/2)$.

We will use that parametrization with respect to the arc-length along the boundary of the polygon. As a result on each side $f(t) = \frac{1}{2}R \cos((\theta_1/2))$. Let us introduce an usual notation here $a_p = R \cos((\theta_1/2))$. We choose \mathbf{u} to be $A_m = (-a_p, \ell/2)$ and as a result $\mathbf{v} = F(\mathbf{u})$ is the midpoint of $A_{2m}A_0$. The coordinates of \mathbf{t} are $(-a_p, t)$, $t \in [0, \ell/2]$ and of $A_{2m} = (R \cos \theta_1, -R \sin \theta_1)$. The equation of $\overline{\mathbf{tF}(\mathbf{t})}$ is $Y = -(t/a_p)X$ and the equation of $\overline{A_{2m}A_0}$ is $X(\sin \theta_1) - Y(1 - \cos \theta_1) - R \sin \theta_1 = 0$, and so the intersection $\mathbf{F}(\mathbf{t})$ has coordinates $x_t = \frac{a_p R \sin \theta_1}{a_p \sin \theta_1 + t(1 - \cos \theta_1)}$ and $y_t = -\frac{tR \sin \theta_1}{a_p \sin \theta_1 + t(1 - \cos \theta_1)}$. Then, $F(t)A_0 = \frac{2tR \sin(\theta_1/2)}{a_p \sin \theta_1 + t(1 - \cos \theta_1)}$. Therefore,

$$H(t) = \frac{1}{2} + \text{Area}(\mathbf{t}BO) - \text{Area}(OA_0\mathbf{F}(\mathbf{t})) = \frac{1}{2}(1 + ta_p - a_p F(t)A_0) \Rightarrow$$

$$H(t) - \frac{1}{2} = \frac{a_p}{2} \left(t - \frac{2tR \sin(\theta_1/2)}{a_p \sin \theta_1 + t(1 - \cos \theta_1)} \right).$$

Hence, we get

$$\mathcal{P} = \frac{1}{4} - 6(2m+1) \frac{a_p^3}{8} \int_0^{\ell/2} \left(t - \frac{2tR \sin(\theta_1/2)}{a_p \sin \theta_1 + t(1 - \cos \theta_1)} \right)^2 dt.$$

Let us change the variable, $t = (\ell/2)s$, $s \in [0, 1]$ and also we will denote $\theta = \theta_1/2$. This implies

$$\mathcal{P} = \frac{1}{4} - 3(2m+1) \frac{R^3 \cos^3 \theta R^3 \sin^3 \theta}{4} \int_0^1 \left(s - \frac{2s \sin \theta}{\cos \theta \sin 2\theta + s \sin \theta (1 - \cos 2\theta)} \right)^2 ds,$$

or

$$\mathcal{P} = \frac{1}{4} - \frac{3}{4(2m+1)^2} \int_0^1 \left(s - \frac{s}{\cos^2 \theta + s \sin^2 \theta} \right)^2 ds.$$

This integration is rather difficult one, but with some help from Maple one obtains

$$\mathcal{P}_m = \frac{1}{4} - \frac{1 + 9a - 9a^2 - a^3 + 6a(1+a) \ln a}{4(2m+1)^2(1-a)^3}, \quad (9)$$

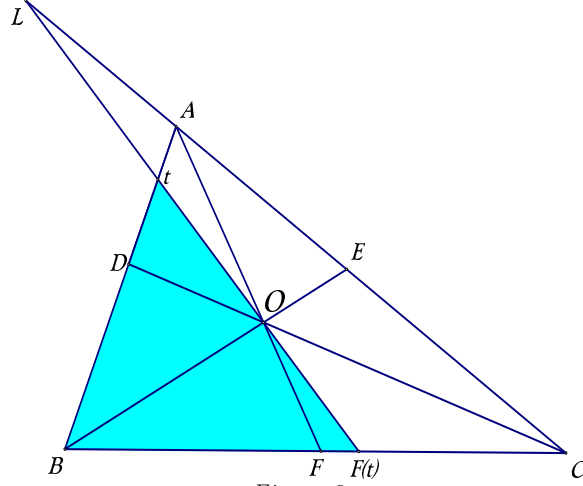
where $a = \cos^2 \theta = \cos^2 \frac{\pi}{2m+1}$.

For a regular pentagon, $m = 2$ and $a = \frac{3+\sqrt{5}}{8}$, one gets a pretty complicated expression for the probability but it is an exact one:

$$\mathcal{P}_5 = \frac{1}{625} \left[(990 + 438\sqrt{5}) \ln(\sqrt{5} - 1) - 30(2 + 3\sqrt{5}) \right] \approx 0.24982224.$$

It is not difficult to check that if $m \rightarrow \infty$, the function in (9) converges indeed to $1/4$.

7 An arbitrary triangle and an arbitrary point in its interior



Let us start with an arbitrary triangle ABC of area equal to 1 (as in Figure 6), and an arbitrary point O in the interior of the triangle. We let $D = F(C)$, $E = F(B)$ and $F(A) = F$. For \overline{uv} we take \overline{AF} and then formula (2) implies:

$$\mathcal{P} = -G(\mathbf{v})^2[3 - 2G(\mathbf{v})] + 6 \int_{\overline{uv}} H(t)[1 - H(t)] f(t) dt.$$

In this situation $G(\mathbf{v})$ is just the area of the triangle AFC . So, we will integrate over the segments \overline{AD} , \overline{DB} and \overline{BF} . Let us denote the distances from O to the sides \overline{BC} , \overline{CA} , and \overline{AB} by d_A , d_B and d_C respectively. Also, we introduce the standard notation for the altitudes of the triangle h_A , h_B and h_C . This gives us the barycentric coordinates of O : $\alpha = \frac{d_A}{h_A}$, $\beta = \frac{d_B}{h_B}$ and $\gamma = \frac{d_C}{h_C}$. We have

$$\alpha + \beta + \gamma = \frac{(BC \cdot d_A)/2}{(BC \cdot h_A)/2} + \frac{(AC \cdot d_B)/2}{(AC \cdot h_B)/2} + \frac{(AB \cdot d_C)/2}{(AB \cdot h_C)/2} = [OBC] + [OAC] + [OAB] = 1,$$

where $[XYZ]$ denotes the area of the triangle XYZ .

The distribution f along the sides is piecewise constant: $\frac{d_C}{2}$ on \overline{AB} , $\frac{d_A}{2}$ on \overline{BC} , and $\frac{d_B}{2}$ on \overline{AC} .

First, we observe that

$$\frac{DA}{DB} = \frac{[ADO]}{[BDO]} = \frac{[ADC]}{[BDC]} = \frac{[AOC]}{[BOC]} = \frac{\beta}{\alpha},$$

and similarly

$$\frac{EA}{EC} = \frac{\gamma}{\alpha}, \quad \text{and} \quad \frac{FB}{FC} = \frac{\gamma}{\beta}.$$

Then,

$$G(\mathbf{v}) = [ABF] = \frac{[ABF]}{[ABC]} = \frac{BF}{BC} = \frac{\gamma}{\beta + \gamma}.$$

In order to calculate the function $H(t)$ for $t \in \overline{AD}$, we denote by x the length of the segment \overline{At} .

LEMMA 7.1. *For $t \in \overline{AD}$, we have*

$$\alpha \frac{tA}{tB} + \gamma \frac{F(t)C}{F(t)B} = \beta. \quad (10)$$

Proof. If $\overline{tF(t)}$ is parallel to \overline{AC} then the two ratios in (10) are equal and equal to $OE/OB = \frac{d_B}{h_b - d_B} = \frac{\beta}{1 - \beta}$, which is precisely (10). If $\overline{tF(t)}$ is not parallel to \overline{AC} , let L be their intersection (see Figure 6). Since $\frac{EA}{EC} = \frac{\gamma}{\alpha}$ we have $\frac{LE - LA}{LC - LE} = \frac{\gamma}{\alpha}$. Solving for LE from this last equality we obtain $LE = \frac{\alpha LA + \gamma LC}{\alpha + \gamma}$. By Menelaus' Theorem in the triangle ABE and the transversal $L - t - O$ we get $\frac{tA}{tB} \frac{OB}{OE} \frac{LE}{LA} = 1$ and solving for $\frac{tA}{tB} = \frac{LA}{LE} \frac{\beta}{1 - \beta}$. Similarly, Menelaus' Theorem in the triangle BOC and the transversal $L - O - F(t)$ gives $\frac{F(t)C}{F(t)B} = \frac{LC}{LE} \frac{\beta}{1 - \beta}$. Then, using the relation about LE that we established earlier we get

$$\alpha \frac{tA}{tB} + \gamma \frac{F(t)C}{F(t)B} = \frac{\alpha LA + \gamma LC}{LE} \frac{\beta}{1 - \beta} = (\alpha + \gamma) \frac{\beta}{1 - \beta} = \beta,$$

which proves (10). ■

Lemma 7.1 allows us to calculate $H(t)$ as a function of x , since the area of the triangle $tBF(t)$ can be written as

$$[tBF(t)] = \frac{[tBF(t)]}{[ABC]} = \frac{tB}{AB} \frac{F(t)B}{BC} = \left(1 - \frac{x}{AB}\right) \left(1 - \frac{F(t)C}{BC}\right).$$

From (10) we obtain

$$\begin{aligned} \frac{F(t)C}{BC} &= \frac{\beta - \alpha \frac{x}{AB - x}}{\beta + \gamma - \alpha \frac{x}{AB - x}} = \frac{\beta - (\alpha + \beta) \frac{x}{AB}}{\beta + \gamma - \frac{x}{AB}} \Rightarrow \\ 1 - \frac{F(t)C}{BC} &= \frac{\gamma(1 - \frac{x}{AB})}{\beta + \gamma - \frac{x}{AB}}. \end{aligned}$$

This implies that $H(t) = \frac{\gamma(1 - \frac{x}{AB})^2}{\beta + \gamma - \frac{x}{AB}}$. In the integral over \overline{AD} we make a substitution $x = ABs$, with $s \in [0, \frac{\beta}{\alpha + \beta}]$ and hence, the contribution of this integral is

$$\int_{\overline{AD}} H(t) [1 - H(t)] f(t) dt = AB(d_C/2) \int_0^{\frac{\beta}{\alpha + \beta}} \frac{\gamma(1 - s)^2}{\beta + \gamma - s} - \frac{\gamma^2(1 - s)^4}{(\beta + \gamma - s)^2} ds.$$

Therefore, this reduces to

$$\int_{\overline{AD}} H(t) [1 - H(t)] f(t) dt = \gamma^2 \int_0^{\frac{\beta}{\alpha+\beta}} \frac{(1-s)^2}{\beta + \gamma - s} - \frac{\gamma(1-s)^4}{(\beta + \gamma - s)^2} ds.$$

If we change the variable again, $\beta + \gamma - s = u$, we obtain

$$I(\alpha, \gamma) = \gamma^2 \int_{\frac{\alpha\gamma}{1-\gamma}}^{1-\alpha} \frac{(\alpha + u)^2}{u} - \frac{\gamma(\alpha + u)^4}{u^2} du.$$

The integrals over \overline{DB} and \overline{BF} are written in a similar way, and so we get an integral formula which for computational purposes is very easy to use even by hand:

$$\mathcal{P} = 6(I(\alpha, \gamma) + I(\beta, \gamma) + I(\beta, \alpha)) - \frac{\gamma^2(3 - 3\alpha - 2\gamma)}{(1 - \alpha)^3}. \quad (11)$$

For instance if $\alpha = \frac{1}{6}$, $\beta = \frac{1}{3}$ and $\gamma = \frac{1}{2}$ then $\mathcal{P} = \frac{1}{27} + \frac{41 \ln 5}{972} + \frac{17 \ln 2}{243} \approx 0.1534$. We used formula (11) to plot this function in 3D over the points of an equilateral triangle in terms of the barycentric coordinates of the points inside the equilateral triangle:

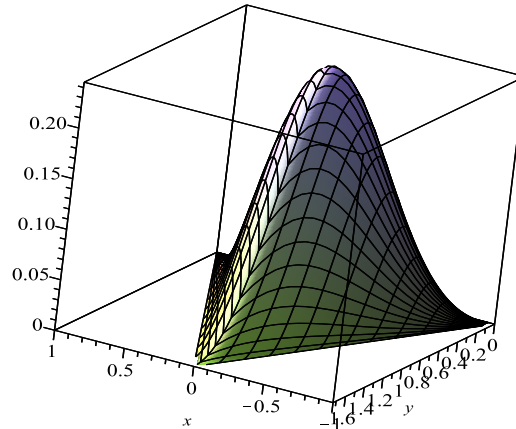


Figure 7

Clearly this graph is consistent to the result that the maximum of the probability in question is attained for $\alpha = \beta = \gamma = \frac{1}{3}$. It is interesting that one can now use (11) to compute \mathcal{P} for O the incenter of triangle, the circumcenter, or the orthocenter. For instance if $O = I$, and the sides of the triangle are a , b and c , then taking $\alpha = \frac{a}{a+b+c}$, $\beta = \frac{b}{a+b+c}$, and $\gamma = \frac{c}{a+b+c}$.

8 The case of a square and an arbitrary point

In what follows we will refer to Figure 6. Without loss of generality we assume the square to be the unit square $S := [0, 1]^2$ and the point $O(u, v)$ with $u, v \in (0, 1)$. The distribution of the points $A' = s(A)$ on the sides is almost uniform. Indeed, let us say the distribution of A' on side UV is given by a continuous function f . Then for $0 \leq a \leq b \leq 1$ we have

$$\int_a^b f(x)dx = \mathbb{P}(A' \in [a, b]) = \text{Area of triangle } \triangle OA'B' = \frac{1}{2}(b-a) \cdot v.$$

Differentiating with respect to b gives $f(b) = \frac{v}{2}$. We denote this distribution corresponding to side UV by f_{UV} and similarly $f_{VW}(y) = \frac{1-u}{2}$, $f_{WZ}(x) = \frac{1-v}{2}$, and finally $f_{ZU}(y) = \frac{u}{2}$. As we have noticed before, we have

$$\int_{\partial S} f(s)ds = \frac{v}{2} + \frac{1-u}{2} + \frac{1-v}{2} + \frac{u}{2} = 1.$$

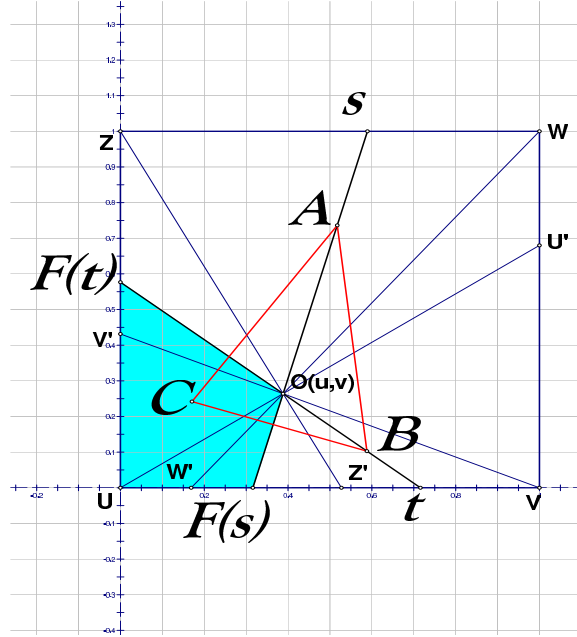


Figure 8

Due to the symmetries of the square, we may assume, without loss of generality that $0 < v < u < \frac{1}{2}$ (the limiting cases $u = v$ and $u = 1/2$, can be obtained by a continuity argument). We let $U' = F(U)$, $V' = F(V)$, $W' = F(W)$ and $Z' = F(Z)$. These points, together with the vertices of the square, divide the boundary ∂S into eight parts. The coordinates of these points are easy to compute: $U' = (1, \frac{v}{u})$, $V' = (0, \frac{v}{1-u})$, $W' = (\frac{u-v}{1-v}, 0)$ and $Z' = (\frac{u}{1-v}, 0)$. So, we have a piecewise definition for F on 8 intervals:

$$F(t) = \begin{cases} (1, \frac{v(1-x)}{u-x}) & \text{if } t = (x, 0), x \in [0, \frac{u-v}{1-v}], \\ (\frac{u-(1-v)x}{v}, 1) & \text{if } t = (x, 0), x \in [\frac{u-v}{1-v}, \frac{u}{1-v}], \\ (0, \frac{vx}{x-u}) & \text{if } t = (x, 0), x \in [\frac{u}{1-v}, 1], \\ (0, \frac{v-uy}{1-u}) & \text{if } t = (1, y), y \in [0, \frac{v}{u}], \end{cases}, \quad F(t) = \begin{cases} (\frac{uy-v}{y-v},) & \text{if } t = (y, 0), y \in [\frac{v}{u}, 1], \\ (\frac{u-vx}{1-v}, 0) & \text{if } t = (x, 1), x \in [0, 1], \\ (0, \frac{uy}{y-v}) & \text{if } t = (0, y), y \in [\frac{v}{1-u}, 1], \\ (0, \frac{v-(1-u)y}{u}) & \text{if } t = (0, y), y \in [0, \frac{v}{1-u}]. \end{cases}$$

For the function H we just need to compute some areas:

$$H(t) = \begin{cases} \frac{v(1-x)^2}{2(u-x)} & \text{if } t = (x, 0), x \in [0, \frac{u-v}{1-v}], \\ \frac{2vx+u-x}{2v} & \text{if } t = (x, 0), x \in [\frac{u-v}{1-v}, \frac{u}{1-v}], \\ \frac{vx^2}{2(x-u)} & \text{if } t = (x, 0), x \in [\frac{u}{1-v}, 1], \\ \frac{v+y(1-2u)}{2(1-u)} & \text{if } t = (1, y), y \in [0, \frac{v}{u}] \end{cases}, \quad H(t) = \begin{cases} \frac{(1-u)y^2}{2(y-v)} & \text{if } t = (y, 0), y \in [\frac{v}{u}, 1], \\ \frac{u+x-2vx}{2(1-v)} & \text{if } t = (x, 0), x \in [0, 1], \\ \frac{uy^2}{2(y-v)} & \text{if } t = (x, 0), x \in [\frac{v}{1-u}, 1], \\ \frac{v-y+2uy}{2u} & \text{if } t = (0, y), y \in [0, \frac{v}{1-u}]. \end{cases}$$

Because there is no obvious point \mathbf{u} for which $G(\mathbf{v}) = \mathbf{1}/2$ we will use formula (2). Then the probability \mathcal{P} is given by four integrals $\mathcal{P} = 6(I_1 + I_2 + I_3 + I_4) - \frac{v^2(3u-v)}{4u^3}$ where

$$I_1 = \int_0^{\frac{u-v}{1-v}} \frac{v^2(1-x)^2}{4(u-x)} \left(1 - \frac{v(1-x)^2}{2(u-x)}\right) dx,$$

$$I_2 = \int_{\frac{u-v}{1-v}}^{\frac{u}{1-v}} \frac{2vx+u-x}{4} \left(1 - \frac{2vx+u-x}{2v}\right) dx,$$

$$I_3 = \int_{\frac{u}{1-v}}^1 \frac{v^2x^2}{4(x-u)} \left(1 - \frac{vx^2}{2(x-u)}\right) dx, \text{ and}$$

$$I_4 = \int_0^{\frac{v}{u}} \frac{v+y(1-2u)}{4} \left(1 - \frac{v+y(1-2u)}{2(1-u)}\right) dy.$$

The answer in general is rather complicated and it is not symmetrical as expected:

$$\mathcal{P} = Q_1(u, v) + Q_2(u, v) \ln \frac{1-u}{u} + Q_3(u, v) \ln \frac{1-v}{v}$$

where Q_i are rational functions in u and v .

For particular situations one can obtain pretty short expressions, for instance, if $u = 1/2$ and $v = 1/4$ we obtain $\mathcal{P} = \frac{5}{48} + \frac{9 \ln 3}{256}$.

In the case $u = v$, the answer is a little more manageable

$$\mathcal{P}_u = \frac{1}{4} - \frac{(1-2u)(1-2u^2)(1+u-6u^3)}{4(1-u)} + 3u^4(1-2u^2) \ln \frac{1-u}{u}, \quad u \in (0, \frac{1}{2}). \quad (12)$$

If $u = v = \frac{1}{3}$ this reduces to $\frac{23}{162} + \frac{7 \ln 2}{243} \approx 0.161942$. We checked these experimentally and in 10^6 trials they check usually with the third decimal. As a curiosity, for $u = v = \frac{1}{1+e}$, the probability is just in terms of e : $\mathcal{P} = \frac{5e^5+6e^4+13e^3+7e^2-6e+1}{e(e+1)^6}$.

9 Circle with a point off center

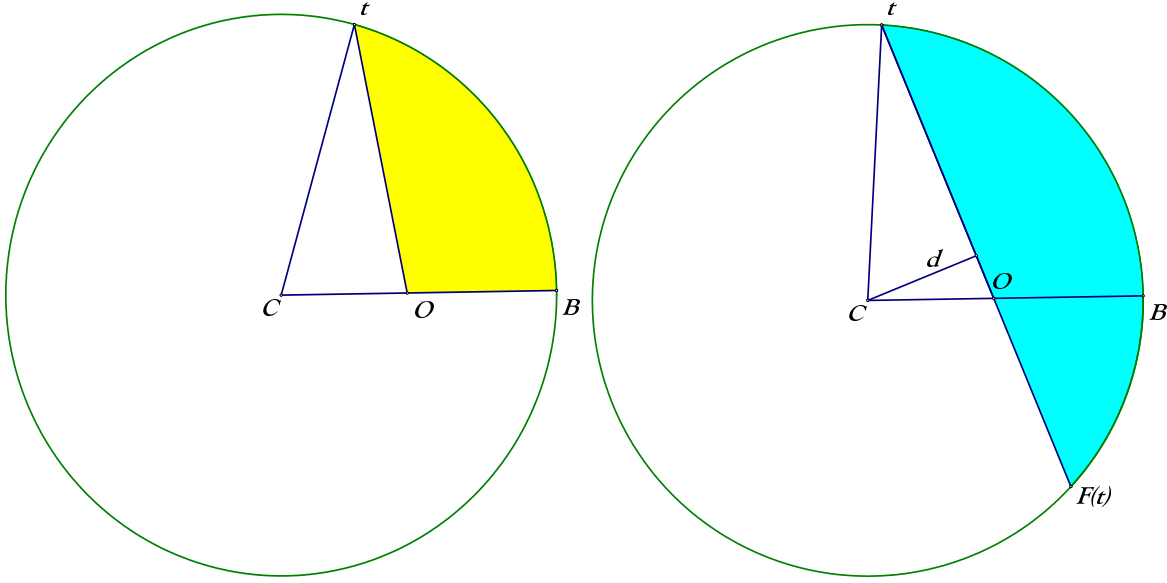


Figure 9

Let us work with a circle centered at the origin of radius $R = \frac{1}{\sqrt{\pi}}$ (Figure 7). The point O has coordinates $(rR, 0)$ with $r \in [0, 1)$. Next, we will check that the function f is given by $f(s) = R^2(1 - r \cos s)/2$ in the sense that

$$\int_0^t f(s) ds = \text{Area}(\widehat{O\hat{B}t}), t \in [0, 2\pi]$$

Indeed the above equality is the same as

$$R^2/2 - R^2 r \sin t/2 = \text{Area}(\text{sector } TCB) - \text{Area}(\triangle TCO) = \text{Area}(\widehat{O\hat{B}t}),$$

which is correct. The function H is given by a well known formula, $1 - H(t) = R^2 \arccos(\frac{d}{R}) - d\sqrt{R^2 - d^2}$, where d is the distance from C to the line $tF(t)$: $d = \frac{R^2 r \sin t}{tO}$. Since $tO = R\sqrt{1 - 2r \cos t + r^2}$ we obtain

$$H(t) = 1 - R^2 \left[\arccos\left(\frac{r \sin t}{\sqrt{1 - 2r \cos t + r^2}}\right) - \frac{r(1 - r \cos t) \sin t}{\sqrt{1 - 2r \cos t + r^2}} \right], t \in [0, \pi].$$

Using (3), the probability in this case is given by

$$\mathcal{P} = \frac{1}{4} - \frac{3}{\pi} \int_0^\pi \left[\frac{1}{2} - \frac{1}{\pi} \arccos\left(\frac{r \sin t}{\sqrt{1 - 2r \cos t + r^2}}\right) + \frac{r(1 - r \cos t) \sin t}{\pi \sqrt{1 - 2r \cos t + r^2}} \right]^2 (1 - r \cos t) dt.$$

It is not clear if this integral can be simplified any further but it is for sure related to the subject of elliptic integrals. For practical purposes it can be used to evaluate the probability numerically. For instance, if $r = 1/2$ one gets $\mathcal{P} \approx 0.2473907871$.

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